# Test 2 - MTH 2410

Dr. Graham-Squire, Fall 2013

12:01
12:23
22 min

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(*signature*)

## DIRECTIONS

1. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
2. Clearly indicate your answer by putting a box around it.
3. Calculators are allowed on all parts of the in-class portion of the test except for the last 4 questions, for which no technology is allowed. Even on questions where technology is allowed, you should still show all of your work. Computers and calculators are allowed on the take home part of the test, and instructions are given on that part.
4. Give all answers in exact form, not decimal form (that is, put $\pi$ instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge.
6. Number of questions = 10. Total Points = 90.

**Calculators allowed**

1. (8 points) TRUE OR FALSE. Circle the correct answer. If false, give a counterexample or explain (briefly) why it is false. If true, no explanation is necessary (though if you are wrong, an explanation can get you some partial credit).

(a) **True or False:** If $f_x(a,b) = f_y(a,b) = 0$ and $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have opposite signs, then $(a,b)$ is a saddle point.

True $d = f_{xx} f_{yy} - (f_{xy})^2$, so if $f_{xx}$ and $f_{yy}$ have opposite signs then $d < 0 \Rightarrow$ saddle.

(b) **True or False:** The arc length of a curve depends on its parametrization.

False. The ~~arc length of a curve~~ parametrization can change the speed of the particle on the curve, but it does not change the shape or length.

(c) **True or False:** If both second partials are defined and continuous at the point $(0,0)$, then $f_{xy}(0,0) = f_{yx}(0,0)$.

True. Mixed partials are always equal.

(d) **True or False:** Every vector-valued function is continuous at every point in its domain.

False.

2. (8 points) Find the curvature $K$ of the curve $\mathbf{r}(t) = t\mathbf{i} + \sqrt{4-t^2}\mathbf{j} + 3\mathbf{k}$.

$$K = \frac{\|r' \times r''\|}{\|r'\|^3} \qquad \text{when } t = 1 \qquad (4-t^2)^{1/2} \Rightarrow \frac{1}{2}(4-t^2)^{-1/2}\cdot(-2t)$$

$$r'(t) = \left\langle 1, \frac{-t}{\sqrt{4-t^2}}, 0 \right\rangle \qquad \|r'(t)\| = \sqrt{1^2 + \frac{1}{3}} = \sqrt{\frac{4}{3}}$$

$$r''(t) = \left\langle 0, \frac{-1(\sqrt{4-t^2}) + t\left(\frac{-t}{\sqrt{4-t^2}}\right)}{4-t^2}, 0 \right\rangle$$

$$= \left\langle 0, \frac{-4+t^2-t^2}{(4-t^2)^{3/2}}, 0 \right\rangle$$

$$= \left\langle 0, \frac{-4}{(4-t^2)^{3/2}}, 0 \right\rangle$$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ \cancel{1} & \frac{-t}{\sqrt{4-t^2}} & 0 \\ 0 & \frac{-4}{(4-t^2)^{3/2}} & 0 \end{vmatrix} = \left\langle 0, 0, \frac{-4}{(4-t^2)^{3/2}} \right\rangle$$

$$\|r' \times r''\| = \frac{4}{(4-t^2)^{3/2}} = \frac{4}{3\sqrt{3}} \qquad \|r'\|^3 = \frac{8}{3\sqrt{3}}$$

$$\Rightarrow K = \frac{4}{(4-t^2)^{3/2}} \cdot \frac{(4-t^2)^{3/2}}{t^3} = \boxed{\frac{4}{t^3}}$$

$$K = \frac{\left(\frac{4}{3\sqrt{3}}\right)}{\left(\frac{8}{3\sqrt{3}}\right)} = \boxed{\frac{1}{2}} \qquad \text{when } t = 1 \text{ get } \quad \frac{4}{1} = \boxed{4}$$

3. (8 points) Calculate the limit: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^3 - y^3}$.

along $x = 0 \Rightarrow \lim_{y \to 0} \frac{0}{-y^3} = 0$

along $y = 0 \Rightarrow \lim_{x \to 0} \frac{0}{x^3} = 0$

along $y = x \Rightarrow \lim_{x \to 0} \frac{x^2}{x^3 - x^3} = \lim_{x \to 0} \frac{x^2}{0} = \infty$, dne

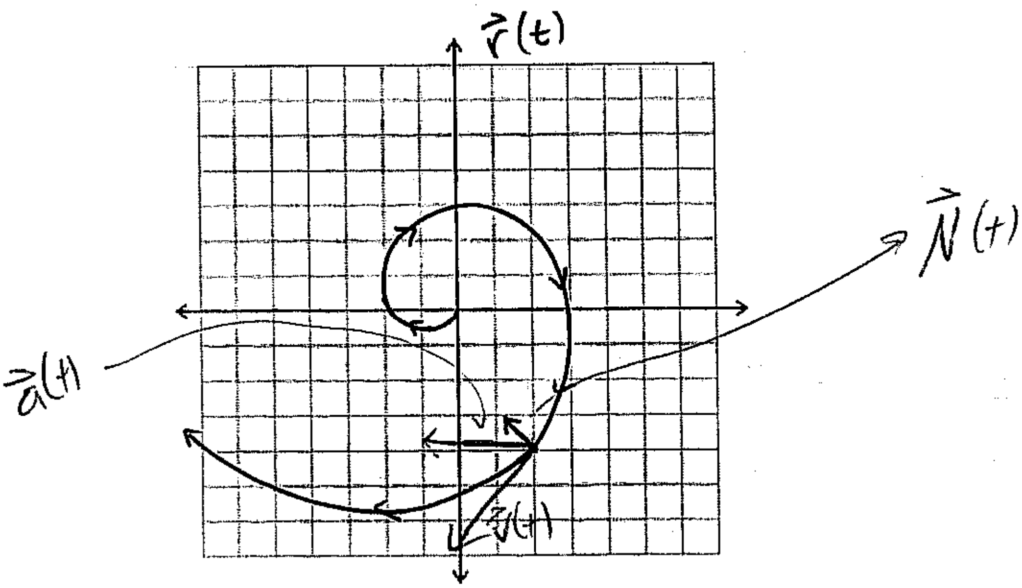
$\Rightarrow$ limit $\boxed{\text{dne}}$

4. (10 points) (a) For the graph of the vector-valued function $\mathbf{r}(t)$ below, at the point (2,-4), sketch an approximation of

(i) the velocity vector $\mathbf{v}(t)$

(ii) the acceleration vector $\mathbf{a}(t)$

(iii) the unit normal vector $\mathbf{N}(t)$




(b) Explain the difference between the velocity $\mathbf{v}(t)$ and the unit tangent vector $\mathbf{T}(t)$.

$\vec{v}(t)$ and $\vec{T}(t)$ point in the same direction, but may have different lengths. $\|T\| = 1$ always, but $\|v\|$ can be any positive number

(c) How would your vectors from part (a) be different (or the same) for the graph of $\mathbf{r}(3t)$?

The $\vec{v}$ and $\vec{a}$ vectors would be lengthened by a factor of 3. $\vec{N}(t)$ would stay the same.

5. (~~6~~ 8 points) Find the equation for the tangent plane to the surface $xy^2 + 3x - z^2 = 8$ at the point (1,-3,2).

$F(x,y,z) = xy^2 + 3x - z^2 - 8$ ✓✓

✓ $F_x = y^2 + 3$

✓ $F_y = 2xy$ at $(1,-3,2) = \langle 12, -6, -4 \rangle$ ✓

✓ $F_z = -2z$

$\Rightarrow$ plane is ✓✓

$$\boxed{12(x-1) - 6(y+3) - 4(z-2) = 0}$$

$$\text{or} \quad \boxed{12x - 6y - 4z = 22}$$

**No Calculator** Name: Key

6. (10 points) The topography of an aquarium is given by the equation $z = \frac{x}{y^2}$. A shaggy mouse nudibranch is at the point with $(x, y)$ coordinates of (-2,1) when it sees a giant purple sea star lumbering over to eat it up.

(a) If the shaggy mouse nudibranch decides to run away from (-2,1) in the steepest direction possible, which direction should it go? Write your answer as a vector.

(b) Suppose instead the nudibranch is at the point (-2,1) and it is facing in the same direction as the positive $x$-axis, and it sees a good rock to hide under exactly 45° to its left (that is, counterclockwise at an angle of $\frac{\pi}{4}$). What will be the slope in that direction?

$z_x = \frac{1}{y^2}$ ✓ $\quad z_y = \frac{-2x}{y^3}$ ✓

$\Rightarrow \nabla z(-2,1) = \langle 1, 4 \rangle$ ✓✓

(a) direction of steepest ascent will be in the direction of the gradient: $\boxed{\langle 1, 4 \rangle}$ 3

(b) Want directional derivative for $\theta = 45°$

$\vec{u} = \langle \cos 45°, \sin 45° \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ 3

$\nabla \cdot \vec{u} = \frac{\sqrt{2}}{2} + \frac{4\sqrt{2}}{2} = \boxed{\frac{5\sqrt{2}}{2}}$

7. (10 points) (a) Calculate the derivative of the vector-valued function

$$\mathbf{r}(t) = \langle \sin(2t), e^t \ln(1+t), \frac{1}{1+t} \rangle$$

$r'(t) = \langle 2\cos(2t), e^t \cdot \frac{1}{1+t} + e^t \ln(1+t), \frac{-1}{(1+t)^2} \rangle$

1.5

4.5

(b) Find the indefinite integral $\int (te^t\mathbf{i} + t\sin(t^2)\mathbf{j} + t^{3/2}\mathbf{k})\, dt$.

5.5

$\int te^t dt = te^t - \int e^t dt = te^t - e^t$

$u = t \quad dv = e^t dt$

$du = dt \quad v = e^t$

$\int t \sin(t^2) dt = \frac{1}{2}\int \sin(u) du = -\frac{1}{2}\cos t^2$

$u = t^2$

$du = 2t\,dt$

$\frac{1}{2} du = t\,dt$

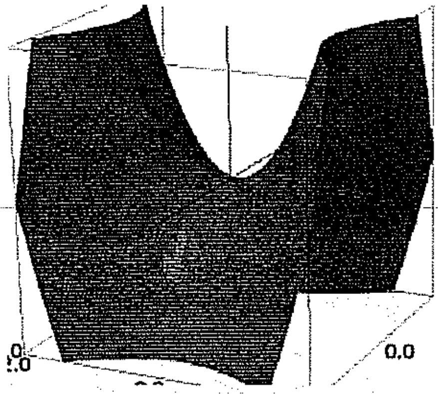
$\Rightarrow \int \langle te^t, t\sin(t^2), t^{3/2} \rangle dt$

$= \langle te^t - e^t, -\frac{1}{2}\cos(t^2), \frac{2t^{5/2}}{5} \rangle + \vec{C}$

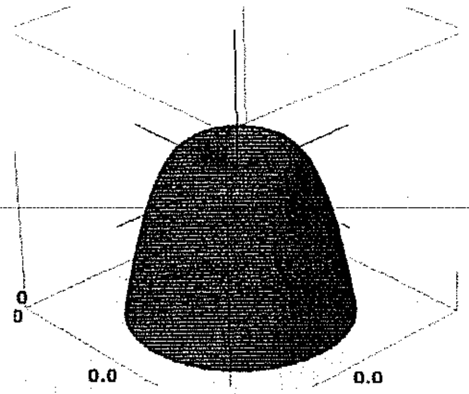
0.5

8. (10 points) Match the function on this page with the graph of its level curves on the next page.

(h)



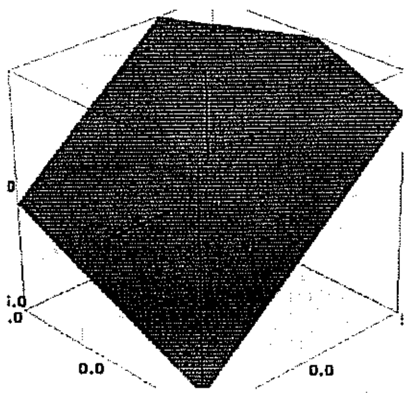

(a)



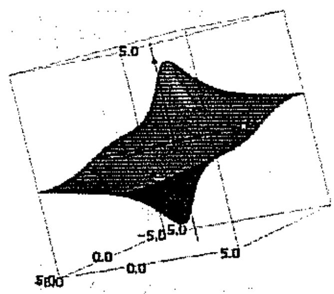

(b)

(j)

(g)

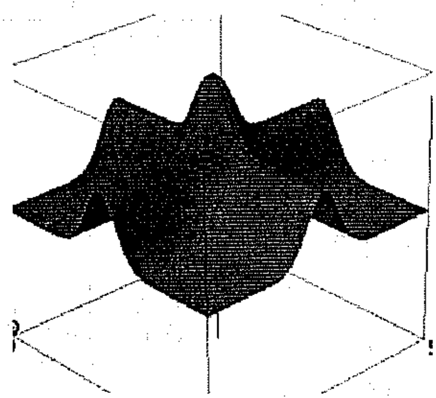



(c)




(d)

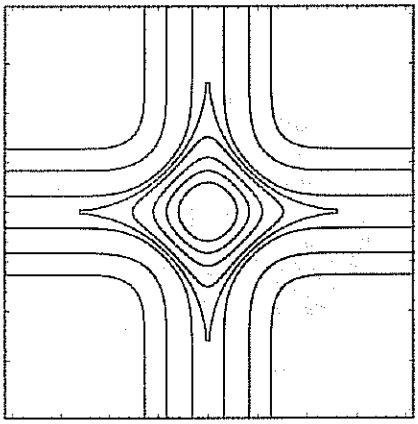
(i)



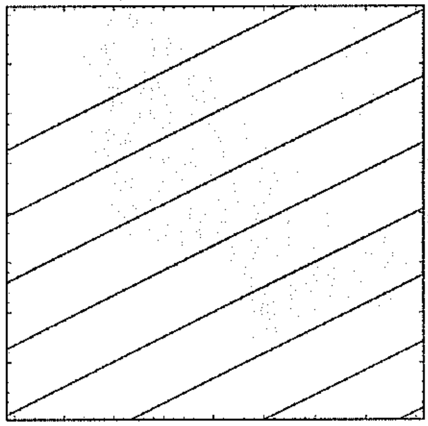
(e)

(f)

(e)



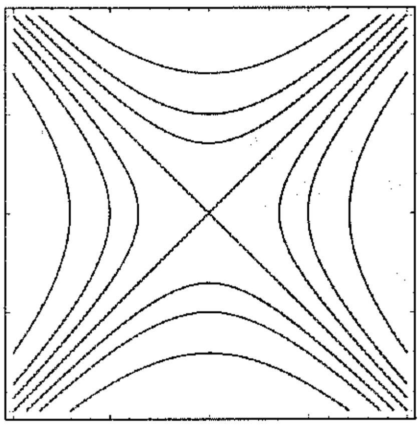
(f)



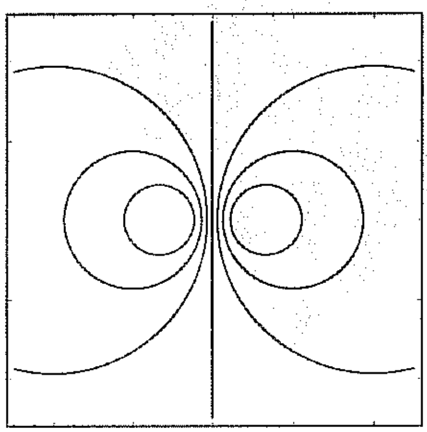
(c)

(g)

(a)

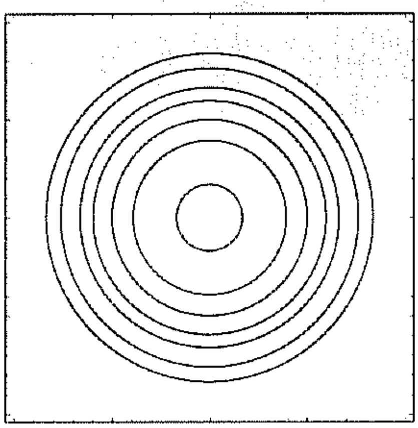


(h)



(d)

(i)



(b)

(j)

9. (8 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the implicitly defined equation

$$z = 3e^x y^2 - 2x^3 y z^2$$

$F(x,y,z) = 3e^x y^2 - 2x^3 y z^2 - z$

$F_x = 3e^x y^2 - 6x^2 y z^2$

$F_y = 6e^x y - 2x^3 z^2$

$F_z = -4x^3 y z - 1$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3e^x y^2 - 6x^2 y z^2)}{-4x^3 y z - 1}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(6e^x y - 2x^3 z^2)}{-4x^3 y z - 1}}$$

**Extra Credit**(2 points) Prove that if $\mathbf{r}(t)$ has constant speed (that is, $||\mathbf{r}'(t)|| = c$), the velocity and acceleration vectors are perpendicular.

$||\vec{r}'(t)|| = c \Rightarrow \vec{r}'(t) \cdot \vec{r}'(t) = c^2$

$\frac{d}{dt}$ both sides to get

$\vec{r}'(t) \cdot \vec{r}''(t) + \vec{r}''(t) \cdot \vec{r}'(t) = 0$

$\Rightarrow 2(\vec{r}'(t) \cdot \vec{r}''(t)) = 0$

$\Rightarrow \vec{r}'(t) \cdot \vec{r}''(t) = 0$

$\boxed{\Rightarrow \vec{v}(t) \perp \vec{a}(t)}$

**Take-home** Name:________________________________

- At some point this weekend you should set aside 30 minutes to work on this question.
- You are allowed to use a calculator and a computer, including programs such as Wolfram Alpha, Sage, Maple or Grapher. All of the calculations can be done by hand (and a calculator, perhaps), and I expect you to completely show your work in order to receive full credit. The computer is only needed to confirm your results and help you check for any mistakes.
- You can use the formula sheet given in class, but you are NOT allowed to discuss this problem with anyone else, nor can you use any other resources such as notes, textbook, etc.
- You ARE allowed to look at the published documents on Sage, however.
- You can use additional paper if you need it.
- The take-home question will be due at the beginning of class on Monday.

I expect the question to take you about 20 minutes, but if it takes you longer that is okay. If it takes you 40 minutes you should stop at 40 and turn in just what you have finished up to that point. Please write down the time you start and end the question.

Start time:

End time:

10. (12 points) Take-home question-

Let $f(x, y) = 3x^2y - 3y + y^3$.

(a) Find and classify all critical points of $f(x, y)$.

(b) Find the absolute extrema of $f(x, y)$ over the rectangular region given by $-2 \leq x \leq 2$ and $0 \leq y \leq 2$.

(a) $f_x = 6xy$

$0 = 6xy$

$\Rightarrow x = 0$

or $y = 0$

$f_y = 3x^2 - 3 + 3y^2$ ✓

$0 = 3(x^2 - 1 + y^2)$

$0 = x^2 + y^2 - 1$ ✓

$\Rightarrow x^2 + y^2 = 1$ ~~correct~~

if $x = 0$ get $y = \pm 1 \Rightarrow (0,1), (0,-1)$

if $y = 0$ get $x = \pm 1 \Rightarrow (1,0), (-1,0)$

✓ are all critical points

✓ $f_{xx} = 6y$

$f_{yy} = 6y$

$f_{xy} = 6x$

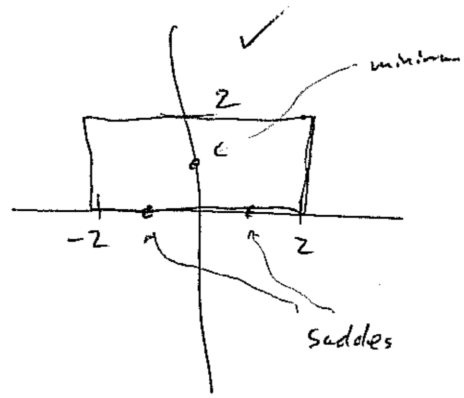
$\Rightarrow d = 36y^2 - 36x^2$

$d$ at $(0,1) = 36 > 0$ $\quad f_{xx} > 0 \Rightarrow$ local <u>minimum</u>

$d$ at $(0,-1) = 36 > 0$ $\quad f_{xx} < 0 \Rightarrow$ local <u>maximum</u>

$$\left.\begin{array}{l} d \text{ at } (1,0) = -36 < 0 \\ d \text{ at } (-1,0) = -36 < 0 \end{array}\right\} \text{Saddle points}$$

(b) $-2 \le x \le 2$ $\qquad 0 \le y \le 2$




Check min value at $(0,1)$:

$f(0,1) = -3 + 1 = -2$

Check boundary: $x = 2 \Rightarrow f(2,y) = 12y - 3y + y^3$

$f'(2,y) = 12 - 3 + 3y^2$

$0 = 9 + 3y^2$

$y^2 = -3$

↳ <u>No</u> crit. pt.

$f(2,0) = 0$ min

$f(2,2) = 26$ max

$\cancel{f(2, \sqrt{\tfrac{1}{3}}) = 9\sqrt{\tfrac{1}{3}} + \tfrac{1}{3\sqrt{3}}}$

$x = -2$ $\qquad f(2,y) = 12y - 3y + y^3$ get same as

$y = 0 \Rightarrow f(x,0) = 0$

$y = 2 \Rightarrow f(x,2) = 6x^2 - 6 + 8 = 6x^2 + 2$

always positive.

minimum at $x = 0$ of 2

max at $x = 2$ (or $-2$)

of 26

$$\Rightarrow \boxed{\begin{array}{l} \text{abs. max of } 26 \\ \text{" min of } -2 \end{array}}$$